

Non t -intersecting families of linear spaces over $\text{GF}(q)$

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Abstract

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Let n, s, t be nonnegative integers with $s \leq t < n$ and let V be an n -dimensional linear space over some finite field $\text{GF}(q)$. Let \mathcal{F} be a family of linear subspaces of V , which satisfies $\dim(F \cap F') < s$ or $\dim(F \cap F') > t$ for all $F, F' \in \mathcal{F}$. In this paper it is shown for $n \geq 9t$ that

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=0}^{s-1} \binom{n}{i}_q + \sum_{i=(n+t+1)/2}^n \binom{n}{i}_q & \text{if } n+t \equiv 1 \pmod{2}, \\ \sum_{i=0}^{s-1} \binom{n}{i}_q + \binom{n-1}{\frac{n+t}{2}}_q + \sum_{i=(n+t+2)/2}^n \binom{n}{i}_q & \text{if } n+t \equiv 0 \pmod{2}. \end{cases}$$

Moreover, all extremal families \mathcal{F} are characterized. This yields a q -analog of a result of Frankl and Füredi (1984), who proved a corresponding result for finite sets.

1. Introduction

Answering a question of Erdős [3] on the maximum size of families \mathcal{F} of finite sets, for which no two members $F, F' \in \mathcal{F}$ intersect in exactly t elements, Frankl and Füredi [6] proved the following.

Theorem 1.1 (Frankl and Füredi [6]). *Let s, t with $s \leq t$ be nonnegative integers. Then there exists a nonnegative integer $n_0(s, t)$ such that for every nonnegative integer $n \geq n_0(s, t)$ it is valid:*

Every family \mathcal{F} of subsets of an n -element set X with $|F \cap F'| < s$ or $|F \cap F'| > t$

for all $F, F' \in \mathcal{F}$ satisfies

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=0}^{s-1} \binom{n}{i} + \sum_{i=(n+t+1)/2}^n \binom{n}{i} & \text{if } n+t \equiv 1 \pmod{2}, \\ \sum_{i=0}^{s-1} \binom{n}{i} + \binom{n-1}{\frac{n+t}{2}} + \sum_{i=(n+t+2)/2}^n \binom{n}{i} & \text{if } n+t \equiv 0 \pmod{2}, \end{cases}$$

and equality holds iff

$$\mathcal{F} = \begin{cases} \left\{ S \subseteq X \mid |S| < s \text{ or } |S| \geq \frac{n+t+1}{2} \right\} & \text{if } n+t \equiv 1 \pmod{2}, \\ \left\{ S \subseteq X \mid |S| < s \text{ or } |S \cap (X \setminus \{x\})| \geq \frac{n+t}{2} \right\} & \text{if } n+t \equiv 0 \pmod{2}, \end{cases}$$

for fixed $x \in X$.

This also generalizes former results of Katona [10], who proved the case $s=0$ and t arbitrary, and of Frankl [5], who does it for $s=t=1$. In this paper we prove the corresponding vector space version of Theorem 1.1. Let V be an n -dimensional linear space over $\text{GF}(q)$. For integers k let $\begin{bmatrix} V \\ k \end{bmatrix}_q$ be the set of k -dimensional linear subspaces of $\text{GF}(q)$. Notice that

$$\left| \begin{bmatrix} V \\ k \end{bmatrix}_q \right| = \binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}.$$

Theorem 1.2. Let s, t with $s \leq t$ be nonnegative integers. Let n with $n \geq 9t$ be a nonnegative integer. Let V be an n -dimensional linear space over $\text{GF}(q)$.

Then for every family

$$\mathcal{F} \subseteq \bigcup_{k=0}^n \begin{bmatrix} V \\ k \end{bmatrix}_q$$

of linear subspaces of V , which satisfies $\dim(F \cap F') < s$ or $\dim(F \cap F') > t$ for all $F, F' \in \mathcal{F}$, it is valid

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=0}^{s-1} \binom{n}{i}_q + \sum_{i=(n+t+1)/2}^n \binom{n}{i}_q & \text{if } n+t \equiv 1 \pmod{2}, \\ \sum_{i=0}^{s-1} \binom{n}{i}_q + \binom{n-1}{\frac{n+t}{2}}_q + \sum_{i=(n+t+2)/2}^n \binom{n}{i}_q & \text{if } n+t \equiv 0 \pmod{2}. \end{cases}$$

Moreover, for $t \geq 1$ equality holds in Theorem 1.2 iff

$$\mathcal{F} = \begin{cases} \bigcup_{i=0}^{s-1} \begin{bmatrix} V \\ i \end{bmatrix}_q \cup \bigcup_{i=(n+t+1)/2}^n \begin{bmatrix} V \\ i \end{bmatrix}_q & \text{if } n+t \equiv 1 \pmod{2}, \\ \bigcup_{i=0}^{s-1} \begin{bmatrix} V \\ i \end{bmatrix}_q \cup \begin{bmatrix} U \\ \frac{n+t}{2} \end{bmatrix}_q \cup \bigcup_{i=(n+t+2)/2}^n \begin{bmatrix} V \\ i \end{bmatrix}_q & \text{if } n+t \equiv 0 \pmod{2}, \end{cases}$$

where $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}_q$ is fixed.

For $s = t = 0$ and $n \equiv 1 \pmod{2}$ the family $\bigcup_{i=(n+1)/2}^n \begin{bmatrix} V \\ i \end{bmatrix}_q$ is the only extremal one, while for $n \equiv 0 \pmod{2}$ there are exactly two types, namely

$$\begin{bmatrix} U \\ \frac{n}{2} \end{bmatrix}_q \cup \bigcup_{i=(n+2)/2}^n \begin{bmatrix} V \\ i \end{bmatrix}_q$$

and

$$\left\{ S \in \begin{bmatrix} V \\ \frac{n}{2} \end{bmatrix}_q \mid v \in \begin{bmatrix} S \\ 1 \end{bmatrix}_q \right\} \cup \bigcup_{i=(n+2)/2}^n \begin{bmatrix} V \\ k \end{bmatrix}_q$$

where $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}_q$ and $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ are fixed.

Theorem 1.2 also generalizes former results of Engel [2], who considered the case $s = t = 0$ and of this author [11], who proved it for $s = 0$ and arbitrary t .

2. Proofs

In order to prove Theorem 1.2 we use the linear algebra approach. This turned out to be an appropriate tool concerning combinatorial extremal problems on families having certain intersection properties, compare Ray-Chaudhuri and Wilson [12], Frankl and Singhi [8] or Frankl and Graham [7].

Let n be a nonnegative integer. Let V be an n -dimensional linear space over $GF(q)$. For families $\mathcal{F}, \mathcal{G} \subseteq \bigcup_{i=0}^n \begin{bmatrix} V \\ i \end{bmatrix}_q$ of linear subspaces of V let $M(\mathcal{F}, \mathcal{G})$ be the $|\mathcal{F}| \times |\mathcal{G}|$ matrix with rows indexed by $F \in \mathcal{F}$ and columns by $G \in \mathcal{G}$ and general entries $m(F, G)$, where

$$m(F, G) = \begin{cases} 1 & \text{if } F \subseteq G, \\ 0 & \text{else.} \end{cases}$$

Moreover, for nonnegative integers $k \leq n$ let

$$\mathcal{F}_k = \left\{ U \in \begin{bmatrix} V \\ k \end{bmatrix}_q \mid U \subseteq F \text{ for some } F \in \mathcal{F} \right\}$$

be the lower k -shadow of \mathcal{F} . The following theorem exhibits connections between non t -intersecting families $\mathcal{F} \subseteq \bigcup_{k=0}^n \begin{bmatrix} V \\ k \end{bmatrix}_q$ and linear algebra. A corresponding result for families of finite sets has been proven by Frankl and Singhi [8].

Theorem 2.1. *Let k, n, t be nonnegative integers with $\max\{t+2, 2t+1\} \leq k \leq n$. Let V be an n -dimensional linear space over $\text{GF}(q)$. Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ be a family of k -dimensional linear subspaces of V such that $\dim(F \cap F') \neq t$ for all $F, F' \in \mathcal{F}$. Then the columns of the matrix $M(\mathcal{F}_{k-t-1}, \mathcal{F})$ are linear independent over the rationals \mathbb{Q} .*

For the proof of 2.1 we use the following number theoretic result of Bang [1], c.f. also Zsigmondy [13], which is also of relevance in the theory of finite linear groups. For primes p and positive integers x let $\text{ord}_p(x)$ denote the order of x in the field \mathbb{Z}_p of residues modulo p , i.e. $\text{ord}_p(x)$ is the least positive integer n with $x^n \equiv 1 \pmod{p}$.

Theorem 2.2 (Bang [1], Zsigmondy [13]). *For every pair (n, x) of positive integers with $n, x \geq 2$, which satisfies (i) $(n, x) \neq (6, 2)$ or (ii) $n = 2$ and $(x+1)$ not a power of 2, there exists a prime p such that $\text{ord}_p(x) = n$.*

Proof of Theorem 2.1. Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ be a family such that $\dim(F \cap F') \neq t$ for all $F, F' \in \mathcal{F}$. At first we consider the case $k-t \geq 3$ and $(k-t, q) \neq (6, 2)$. By Theorem 2.2 there exists a prime p with $\text{ord}_p(q) = k-t$. Suppose to the contrary that the columns of $M(\mathcal{F}_{k-t-1}, \mathcal{F})$ are linear dependent over \mathbb{Q} . Then there exist integers α_F , not all zero, with $F \in \mathcal{F}$ such that

$$\sum_{\substack{Y \subseteq F \\ F \in \mathcal{F}}} \alpha_F = 0 \quad \text{for all } Y \in \begin{bmatrix} Y \\ k-t-1 \end{bmatrix}_q. \quad (2.1.1)$$

Assume w.l.o.g. that $\alpha_{F^*} \not\equiv 0 \pmod{p}$ for $F^* \in \mathcal{F}$ fixed. Let j be a nonnegative integer with $j \leq k-t-1$. By (2.1.1) for every $X \in \mathcal{F}_j$ it is valid

$$\sum_{\substack{X \subseteq F \\ F \in \mathcal{F}}} \alpha_F = \frac{1}{\binom{k-j}{k-t-1-j}_q} \sum_{\substack{X \subseteq Y \\ Y \in \begin{bmatrix} V \\ k-t-1 \end{bmatrix}_q}} \sum_{\substack{Y \subseteq F \\ F \in \mathcal{F}}} \alpha_F = 0.$$

Summing up these equations for $X \in \begin{bmatrix} F^* \\ j \end{bmatrix}_q$ yields

$$\sum_{X \in \begin{bmatrix} F^* \\ j \end{bmatrix}_q} \sum_{\substack{X \subseteq F \\ F \in \mathcal{F}}} \alpha_F = \sum_{F \in \mathcal{F}} \alpha_F \cdot \binom{\dim(F \cap F^*)}{j}_q = 0. \quad (2.1.2)$$

Let $g: \mathbb{N} \rightarrow \mathbb{Q}$ be a function, where $\mathbb{N} = \{0, 1, 2, \dots\}$, defined by $g(x) = \prod_{i=t+1}^{k-1} (q^{x-i} - 1)$. Obviously, it is $g(x) = 0$ for integers $t < x < k$. For integers x with $0 \leq x < t$ we have by choice of the prime p that

$$g(x) = \prod_{i=t+1}^{k-1} (q^{x-i} - 1) = (-1)^{k-t-1} \cdot \prod_{i=t+1}^{k-1} \frac{q^{i-x} - 1}{q^{i-x}} \equiv 0 \pmod{p},$$

since $k \geq 2t + 1$. On the other hand,

$$g(k) = \prod_{i=t+1}^{k-1} (q^{k-i} - 1) = \prod_{i=1}^{k-t-1} (q^i - 1) \not\equiv 0 \pmod{p},$$

since $\text{ord}_p(q) = k - t$. Choose rationals $\lambda_0, \lambda_1, \dots, \lambda_{k-t-1} \in \mathbb{Q}$ such that

$$g(x) = \sum_{j=0}^{k-t-1} \lambda_j \binom{x}{j}_q$$

for all $x \in \mathbb{N}$. Multiplication of (2.1.2) by λ_j for $j = 0, 1, \dots, k - t - 1$ and addition of the resulting equations yields

$$\begin{aligned} 0 &= \sum_{j=0}^{k-t-1} \lambda_j \sum_{F \in \mathcal{F}} \alpha_F \cdot \binom{\dim(F \cap F^*)}{j}_q \\ &= \sum_{F \in \mathcal{F}} \alpha_F \sum_{j=0}^{k-t-1} \lambda_j \binom{\dim(F \cap F^*)}{j}_q \\ &= \alpha_{F^*} \cdot g(k) \pmod{p} \not\equiv 0 \pmod{p}, \end{aligned} \tag{2.1.3}$$

since $\alpha_{F^*} \cdot g(k) \not\equiv 0 \pmod{p}$, which gives a contradiction.

Now let $k - t = 6$ and $q = 2$. Then for integers x with $0 \leq x < t$ it is valid

$$g(x) = (-1)^5 \cdot \prod_{i=t+1}^{k-1} \frac{2^{i-x} - 1}{2^{i-x}} \equiv -(2^6 - 1) \cdot \prod_{\substack{i=t+1 \\ i \neq k-t+x}}^{k-1} \frac{2^{i-x} - 1}{2^{i-x}} \equiv 0 \pmod{27},$$

as $2^{2i} - 1 \equiv 0 \pmod{3}$ for all integers i , whereas

$$g(k) = \prod_{i=1}^5 (2^i - 1) \not\equiv 0 \pmod{27}.$$

By choosing $F^* \in \mathcal{F}$ with w.l.o.g. $\alpha_{F^*} \not\equiv 0 \pmod{3}$ we obtain by (2.1.3.)

$$0 = \sum_{F \in \mathcal{F}} \alpha_F \cdot \sum_{j=0}^5 \lambda_j \binom{\dim(F \cap F^*)}{j}_q \equiv \alpha_{F^*} \cdot g(k) \pmod{27} \not\equiv 0 \pmod{27},$$

which is again a contradiction.

Now let $k - t = 2$ and $q = 2^m - 1$ for some integer $m \geq 2$. For integers x with $0 \leq x < t$ it follows

$$g(x) = -\frac{(2^m - 1)^{t+1-x} - 1}{(2^m - 1)^{t+1-x}} \equiv 0 \pmod{4}$$

whereas

$$g(k) = 2^m - 2 \equiv 2 \pmod{4}.$$

Choose $F^* \in \mathcal{F}$ with w.l.o.g. $\alpha_{F^*} \not\equiv 0 \pmod{2}$ and apply (2.1.3). Thus

$$0 = \sum_{F \in \mathcal{F}} \alpha_F \cdot \sum_{j=0}^l \lambda_j \binom{\dim(F \cap F^*)}{j}_q \equiv \alpha_{F^*} \cdot g(k) \pmod{4} \not\equiv 0 \pmod{4},$$

which is again a contradiction and, finally, proves the theorem. \square

The next result, which is due to Frankl and Graham [7], exhibits the connection between the linear independence of the columns of incidence matrices $M(\mathcal{F}_i, \mathcal{F})$ and the shadows of families \mathcal{F} .

Theorem 2.3 (Frankl and Graham [7]). *Let i, j, k, n be nonnegative integers with $i \leq j \leq k \leq n$. Let V be an n -dimensional linear space over $\text{GF}(q)$. Let $\mathcal{F} \subseteq \binom{V}{k}_q$ be a family of k -dimensional linear subspaces of V such that the columns of $M(\mathcal{F}_i, \mathcal{F})$ are linear independent over \mathbb{Q} . Then*

$$|\mathcal{F}_j| \geq |\mathcal{F}| \cdot \frac{\binom{k+i}{j}_q}{\binom{k+i}{k}_q}.$$

For the proof of Theorem 1.2 we use the following q -analog of the Erdős–Ko–Rado theorem [4].

Theorem 2.4 (Frankl and Wilson [9]). *Let k, n, t be nonnegative integers with $t \leq k \leq n$. Let V be an n -dimensional linear space over $\text{GF}(q)$.*

Let $\mathcal{F} \subseteq \binom{V}{k}_q$ be a family of k -dimensional linear subspaces of V such that $\dim(F \cap F') \geq t$ for all $F, F' \in \mathcal{F}$.

Then

$$|\mathcal{F}| \leq \begin{cases} \binom{n-t}{k-t}_q & \text{if } n \geq 2k, \\ \binom{2k-t}{k}_q & \text{if } 2k-t \leq n < 2k, \\ \binom{n}{k}_q & \text{if } n < 2k-t. \end{cases}$$

Moreover, for $n \geq 2k+1$ the unique extremal families are $\mathcal{F}(T) = \{U \in \binom{V}{k}_q \mid T \subseteq U\}$ where $T \in \binom{V}{t}_q$ is fixed, and for $2k-t < n < 2k$ these are $\binom{S}{k}_q$ for some fixed $S \in \binom{V}{2k-t}_q$. If $n = 2k$, there are at least two types of extremal families, namely $\mathcal{F}(T)$ and $\binom{S}{k}_q$ and for $t = 1$ these are the only ones, c.f. [9].

Now we come to the proof of the main theorem. For the case $s = t = 0$ a q -analog of Katona's theorem [10] on finite sets has been given by Engel [2]. A full q -analog, i.e. $s = 0$ and t arbitrary was given by this author.

Theorem 2.5 (Lefmann [11]). *Let n, t be nonnegative integers with $t \leq n$. Let V be an n -dimensional linear space over $\text{GF}(q)$.*

Let $\mathcal{F} \subseteq \bigcup_{k=0}^n [k]_q^V$ be a family of linear subspaces of V such that $\dim(F \cap F') > t$ for all $F, F' \in \mathcal{F}$. Then

$$|\mathcal{F}| \leq \begin{cases} \sum_{k=(n+t+1)/2}^n \binom{n}{k}_q & \text{if } n+t \equiv 1 \pmod{2}, \\ \left\lfloor \frac{n-1}{2} \right\rfloor + \sum_{k=(n+t+2)/2}^n \binom{n}{k}_q & \text{if } n+t \equiv 0 \pmod{2}. \end{cases}$$

Moreover, the extremal families are exactly those mentioned in the remarks following Theorem 1.2 for $s = 0$.

Proof of Theorem 1.2. By Theorem 2.5 we can assume that n, s, t are positive integers with $s \leq t$. Moreover, let $9t \leq n$. Let $\mathcal{F} \subseteq \bigcup_{k=0}^n [k]_q^V$ be a family of linear subspaces of an n -dimensional linear space over $\text{GF}(q)$ such that $\dim(F \cap F') < s$ or $\dim(F \cap F') > t$ for all $F, F' \in \mathcal{F}$. For $k = 0, 1, \dots, n$ put $\mathcal{F}(k) = \mathcal{F} \cap [k]_q^V$. We define for families $\mathcal{H} \subseteq [k]_q^V$ corresponding families $\mathcal{C}(\mathcal{H}) \subseteq [n-k]_q^V$ of complements as follows. Let $\mathcal{G}_k = (\mathcal{A}_k \cup \mathcal{B}_k, \mathcal{E}_k)$ be bipartite graphs with vertex-sets $\mathcal{A}_k = [k]_q^V$ and $\mathcal{B}_k = [n-k]_q^V$ and edge-sets

$$\mathcal{E}_k = \{ \{A, B\} \mid A \in \mathcal{A}_k, B \in \mathcal{B}_k \text{ and } \dim(A \cap B) = 0 \}.$$

For a subset $\mathcal{C}_k \subseteq \mathcal{A}_k \cup \mathcal{B}_k$ let

$$\Gamma(\mathcal{C}_k) = \{X \in \mathcal{A}_k \cup \mathcal{B}_k \mid \{C, X\} \in \mathcal{E}_k \text{ for some } C \in \mathcal{C}_k\}$$

be the set of neighbours of \mathcal{C}_k in \mathcal{G}_k . Since each element $C \in \mathcal{A}_k \cup \mathcal{B}_k$ has exactly $q^{k \cdot (n-k)}$ neighbours in \mathcal{G}_k , it follows

$$|\Gamma(\mathcal{C}_k)| \geq \frac{|\mathcal{C}_k| \cdot q^{k \cdot (n-k)}}{q^{(n-k) \cdot k}} = |\mathcal{C}_k|.$$

By the well-known marriage theorem there exist one-to-one mappings $f_k: [k]_q^V \rightarrow [n-k]_q^V$ with $\dim(U \cap f_k(U)) = 0$ for all $U \in [k]_q^V$. Put $\mathcal{C}(\mathcal{H}) = \{f_k(U) \mid U \in \mathcal{H}\}$.

Fact.

$$\frac{q^k - 1}{q^{k-t} - 1} \cdot |\mathcal{F}(k)| + |\mathcal{F}(n-k+t)| \leq \binom{n}{n-k+t}_q$$

for all $2t+1 \leq k < \frac{n+t}{2}$. (2.6.1)

Proof of Fact. We show first $\mathcal{F}(k)_{k-t} \cap \mathcal{C}(\mathcal{F}(n-k+t)) = \emptyset$. Suppose that there exist $F \in \mathcal{F}(n-k+t)$ and $G \in \mathcal{F}(k)$ with $f_{n-k+t}(F) \subseteq G$. Then

$$\dim(F \cap G) = \dim F + \dim(G) - \dim(F \vee G) \geq t,$$

where $F \vee G$ denotes the span of F and G . As F and $f_{n-k+t}(F)$ are complementary, this implies $\dim(F \cap G) = t$, contradicting the assumption on \mathcal{F} . Thus

$$|\mathcal{F}(k)_{k-t}| + |\mathcal{F}(n-k+t)| \leq \binom{n}{k-t}.$$

By Theorem 2.1 and 2.3 it is

$$|\mathcal{F}(k)_{k-t}| \geq \frac{q^k - 1}{q^{k-t} - 1} \cdot |\mathcal{F}(k)|$$

which proves the fact. \square

Now we consider the case where $n+t \equiv 0 \pmod{2}$ and $k = (n+t)/2$. Let $\langle \cdot, \cdot \rangle: V^2 \rightarrow \text{GF}(q)$ be a scalar product. For linear subspaces U of V let

$$U^\perp = \{v \in V \mid \langle u, v \rangle = 0 \text{ for all } u \in U\}$$

be the orthogonal complement of U . For families $\mathcal{F} \subseteq \bigcup_{k=0}^n \binom{V}{k}_q$ let $\mathcal{F}^\perp = \{F^\perp \mid F \in \mathcal{F}\}$ be the corresponding family of orthogonal complements. Let $F_0, F_1 \in \mathcal{F}((n+t)/2)$. Then it is valid

$$\begin{aligned} \dim(F_0^\perp \cap F_1^\perp) &= n - (\dim F_0 + \dim F_1 - \dim(F_0 \cap F_1)) \\ &= \dim(F_0 \cap F_1) - t \neq 0. \end{aligned}$$

Thus, since

$$\left| \mathcal{F}\left(\frac{n+t}{2}\right)^\perp \right| = \left| \mathcal{F}\left(\frac{n+t}{2}\right) \right|,$$

Theorem 2.4 yields

$$\left| \mathcal{F}\left(\frac{n+t}{2}\right) \right| \leq \binom{n-1}{\frac{n+t}{2}}_q. \quad (2.6.2)$$

Notice that

$$\begin{aligned} |\mathcal{F}(k)| &= 0 & \text{for } s \leq k \leq t, \\ |\mathcal{F}(k)| &\leq \binom{n}{k}_q & \text{for } 0 \leq k < s \text{ or } n-t \leq k \leq n. \end{aligned} \quad (2.6.3)$$

Suppose now that there exists an integer l with $t < l \leq 2t$ and $\mathcal{F}(l) \neq \emptyset$. Put $x_l = \lfloor (n+t)/2 \rfloor + l - t$, then there are

$$\sum_{i=s}^l \binom{l}{i}_q \cdot \binom{n-l}{x_l-i}_q \cdot q^{(x_l-i)(l-i)}$$

many x_l -dimensional linear subspaces U of V with $s \leq \dim(F \cap U) \leq t$, which implies

$$|\mathcal{F}(x_l)| \leq \binom{n}{x_l}_q - \sum_{i=s}^t \binom{l}{i}_q \cdot \binom{n-l}{x_l-i}_q \cdot q^{(x_l-i)(l-i)}.$$

Since $x_l \leq n - t - 1$ for $n \geq 2l + t + 2$ we can apply (2.6.1) and obtain

$$|\mathcal{F}(n+t-x_l)| \leq \left(\binom{n}{x_l}_q - |\mathcal{F}(x_l)| \right) \cdot \frac{q^{n-x_l} - 1}{q^{n+t-x_l} - 1}.$$

Consequently

$$\begin{aligned} & |\mathcal{F}(x_l)| + |\mathcal{F}(n+t-x_l)| \\ & \leq \binom{n}{x_l}_q - \frac{q^{n+t-x_l} - q^{n-x_l}}{q^{n+t-x_l} - 1} \cdot \sum_{i=s}^t \binom{l}{i}_q \cdot \binom{n-l}{x_l-i}_q \cdot q^{(x_l-i)(l-i)}. \quad \square \quad (2.6.4) \end{aligned}$$

Let $L = \{l \in \mathbb{N} \mid t < l \leq 2t \text{ and } \mathcal{F}(l) \neq \emptyset\}$ and $X_L = \{x_l \mid l \in L\}$ and

$$c(n, t) = \begin{cases} \sum_{i=0}^{s-1} \binom{n}{i}_q + \sum_{i=(n+t+1)/2}^n \binom{n}{i}_q & \text{if } n+t \equiv 1 \pmod{2} \\ \sum_{i=0}^{s-1} \binom{n}{i}_q + \left(\frac{n-1}{2} \right)_q + \sum_{i=(n+t+2)/2}^n \binom{n}{i}_q & \text{if } n+t \equiv 0 \pmod{2}. \end{cases}$$

Addition of (2.6.1) for $2t+1 \leq k < (n+t)/2$ and $n+t-k \notin X_L$, and possibly (2.6.2) for $n+t \equiv 0 \pmod{2}$ and (2.6.3) for $0 \leq k \leq t$ and $n-t \leq k \leq n$ and (2.6.4) for $l \in L$ yields

$$\begin{aligned} |\mathcal{F}| &= \sum_{k=0}^n |\mathcal{F}(k)| \leq c(n, t) - \sum_{\substack{k=2t+1 \\ n-k+t \notin X_L}}^{\lfloor (n+t-1)/2 \rfloor} \frac{q^k - q^{k-t}}{q^{k-t} - 1} \cdot |\mathcal{F}(k)| \\ &\quad - \sum_{l \in L} \frac{q^{n+t-x_l} - q^{n-x_l}}{q^{n+t-x_l} - 1} \cdot \sum_{i=s}^t \binom{l}{i}_q \cdot \binom{n-l}{x_l-i}_q \cdot q^{(x_l-i)(l-i)} + \sum_{l \in L} \binom{n}{l}_q. \end{aligned}$$

Using the fact that $\binom{l}{i}_q \cdot \binom{n-l}{x_l-i}_q \cdot q^{(x_l-i)(l-i)}$ is decreasing for $s \leq i \leq t$ and $n \geq 7t - 4s + 2$ and the simple inequality

$$q^{b-a} < \frac{q^b - 1}{q^a - 1} < q^{b-a+1} \quad \text{for } b \geq a \geq 1$$

one obtains

$$\begin{aligned}
 |\mathcal{F}| &\leq c(n, t) - \sum_{\substack{k=2t+1 \\ n-k-t \notin X_l}}^{\lfloor (n+t-1)/2 \rfloor} \frac{q^k - q^{k-t}}{q^{k-t} - 1} \cdot |\mathcal{F}(k)| \\
 &\quad - \sum_{l \in L} ((t-s+1) \cdot q^{t(l-n-t+x_l)+nx_l-x_l^2-1} - q^{(n-l+1)l}) \\
 &\leq c(n, t) - \sum_{\substack{k=2t+1 \\ n-k-t \notin X_l}}^{\lfloor (n+t-1)/2 \rfloor} \frac{q^k - q^{k-t}}{q^{k-t} - 1} \cdot |\mathcal{F}(k)|
 \end{aligned}$$

for $n \geq 9t$.

Thus $|\mathcal{F}| = c(n, t)$ implies $\mathcal{F}(k) = \emptyset$ for all integers k with $s \leq k \leq \lfloor (n+t-1)/2 \rfloor$. Concerning the extremal families \mathcal{F} , i.e. $|\mathcal{F}| = c(n, t)$, we have to consider only the case $n+t \equiv 0 \pmod{2}$ and $k = (n+t)/2$. Let

$$\left| \mathcal{F}\left(\frac{n+t}{2}\right) \right| = \binom{n-1}{\frac{n+t}{2}}_q.$$

For all $F, F' \in \mathcal{F}((n+t)/2)$ we have $\dim(F \cap F') \geq t+1$. By Theorem 2.4 and the remarks following it we have

$$\mathcal{F}\left(\frac{n+t}{2}\right) = \left[\begin{matrix} S \\ \frac{n+t}{2} \end{matrix} \right]_q$$

for some fixed $S \in \left[\begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right]_q$. This proves the theorem. \square

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